D. Summary of similarity laws

The symmetry of the nonaffine similarity laws derived above becomes apparent when comparing Eqs. (10) and (13), Eqs. (11) and (12), Eqs. (14) and (15). Eqs. (10) and (13) state that the force coefficient (with the appropriate reference length and Newtonian constants) in the direction of the complementary parameter ξ less the maximum width (distance normal to the free-stream vector) is invariant. Eqs. (11) and (12) state that the sum of the force coefficients normal to the complementary parameter ξ (with the appropriate reference lengths and Newtonian constants) is equal to the maximum value of this parameter, $\bar{\xi}$. The symmetry of Eqs. (14) and (15) is obvious.

Complementary Configurations

To find complementary configurations for which the similarity laws of the previous section apply, Eq. (4) may be rewritten as follows:

$$y_1'(\xi) = 1/y_0'(\xi)$$

Integrating this equation for configurations complementary with respect to x, there results

$$y_1(x) = \int_0^x \frac{dx}{y_0'(x)} \tag{16}$$

and for configurations complementary with respect to y

$$x_1(y) = \int_0^y y_0'(y)dy \tag{17}$$

For configurations complementary with respect to s, it can be shown that the following transformation is effected

$$(x_0, y_0) \rightarrow (y_1, x_1) \tag{18}$$

Examples of complementary configurations obtained by applying Eqs. (16–18) to a flat-topped parabolic body with a leading-edge angle of 50° and a fineness ratio of 1.5 are shown in Fig. 2. Note that the transformed bodies are concave; whereas the original is convex. This is inherent in single transformations of complementary configurations.

Multiply-complementary configurations arise when more than one transformation is effected. For even-multiple configurations, the transformations are convex-to-convex and concave-to-concave; and for odd-multiple configurations, the transformations are convex-to-concave and concave-to-convex. (Since Newtonian theory may have limited application to concave bodies, the even multiplies for convex bodies are of practical interest.) Note that even-multiple configurations have the same leading-edge angle. Therefore, their Newtonian constants may be taken to be equal; i.e., $k_0 = k_2 = k_4 \ldots$

Examples of multiply-complementary configurations are shown in Fig. 3 for a power-law body with exponent $\frac{3}{4}$. Note that the complementary configurations are also power-law bodies. In general (with some exceptions), power-law bodies remain power-law bodies under complementary transformations.

Conclusions

Nonaffine similarity laws and transformations have been derived, subject to the limitations of Newtonian impact theory. Applying these transformations to different configurations, the lift and drag coefficients of complementary configurations can then be related through the appropriate similarity laws. The results of this paper may be extended to three-dimensional configurations and generalized to configurations which are not complementary (i.e., $\theta_0 + \theta_1 \neq \pi/2$).

References

[1] Shapiro, A. H., *The Dynamics and Thermodynamics of Compressible Fluid Flow*, Vols. I & II, Ronald Press Co., New York, 1954.

[2] Von Kármán, T., "Similarity Law of Transonic Flow," *Journal of Mathematics and Physics*, Vol. 26, 1947, p. 182.

[3] Spreiter, J. R., "On Alternative Forms for the Basic Equations of Transonic Flow Theory," *Journal of the Aeronautical Sciences*, Vol. 21, No. 1, 1954, p. 70.

[4] Tsien, H. S., "Similarity Laws of Hypersonic Flows," *Journal of Mathematics and Physics*, Vol. 25, No. 3, 1946, p. 247.

[5] Hamaker, F. M., Neice, S. E., and Eggers, A. J., Jr., "The Similarity Law for Hypersonic Flow about Slender Three-Dimensional Shapes," TN 2443, 1951, NACA.

A Molecular Model for Tangential Momentum Accommodation

Robert H. Notter,[*] Yukio J. Kaku,[†]
and Norman F. Sather[‡]
University of Washington, Seattle, Wash.

The effectiveness of tangential momentum transfer from a flowing gas to a boundary surface is generally described in terms of a tangential momentum accommodation coefficient σ defined as the fraction of the flux of tangential momentum transmitted to the surface by molecular collisions or, equivalently, the fraction of molecules that are diffusely reflected from the surface. Values of σ for a number of gas-solid systems have been obtained from measurements of slip velocity and surface drag.[1–4] The purpose of this Note is to report calculated values of σ for an idealized, monocrystalline model of the gas-solid interface and to compare these with the experimental values. From this comparison the effect of the polycrystallinity of real surfaces on tangential momentum transport can be inferred. It is also shown that the values of σ obtained from the slip velocity for the model system are different, in general, from those determined from the calculated surface drag. This suggests that experimental values of the accommodation coefficient for certain real surfaces may also depend on which of the two quantities is measured. The results of the study are most applicable to the aerodynamically interesting case of high-speed, low-density gas flows, although at least qualitative inferences can be drawn for other flow systems.

Consider a rarefied gas confined to the space $y > 0$ by an infinite flat plate at $y = 0$. The plate is stationary and the gas flows parallel to it with a velocity $\mathbf{u} = \hat{\mathbf{x}}u(y,t)$. We assume that far from the plate the flow is steady and uniform; thus

$$\lim_{y \to \infty} u(y,t) = U$$

a constant. The molecular state of the gas is given by $f(\mathbf{c},y,t)$, the local distribution function for molecular velocities $\mathbf{c}$, and the macroscopic properties of the gas are moments of f. In particular, the gas velocity at the surface and the interfacial momentum flux are given by

$$\mathbf{u}_s = n_s^{-1} \int \mathbf{c} f_s d\mathbf{c} \tag{1a}$$

and

$$\boldsymbol{p}_s = \int m(\mathbf{c} - \mathbf{u}_s)(\mathbf{c} - \mathbf{u}_s) f_s d\mathbf{c} \tag{1b}$$

Received June 1, 1970; revision received July 9, 1970. This research was supported by a grant from the National Science Foundation.

[*] NIH Postdoctoral Fellow, Department of Chemical Engineering.

[†] Research Assistant, Department of Chemical Engineering; presently Research Engineer, Lockheed Missiles and Space Company, Sunnyvale, Calif.

[‡] Associate Professor, Department of Chemical Engineering.

where m is the molecular mass, f_s is the surface distribution function $f(\mathbf{c},0,t)$, and n_s is the gas density at the surface, i.e., $n_s = \int f_s d\mathbf{c}$.

The surface distribution function needed for the evaluation of Eqs. (1) is obtained from the Liouville theorem which, for free-molecule flow, requires that

$$f_s(\mathbf{c}) = \begin{cases} f^{[0]}(\mathbf{c}) \text{ for } c_y < 0 \\ f^{[0]}(\mathbf{c}') \text{ for } c_y > 0 \end{cases} \tag{2}$$

where $\mathbf{c}'$ is the precollisional velocity of a molecule that leaves with velocity $\mathbf{c}$ after colliding with the surface and $f^{[0]}(\mathbf{c})$ is the Maxwellian distribution function for the incident stream with uniform density n_0, temperature T_0, and velocity $\mathbf{u}_0 = \hat{\mathbf{x}}U$.

The calculation of the slip velocity and surface stress from Eqs. (1) and (2) depends on the surface collision equations which give $\mathbf{c}'$ as a function of $\mathbf{c}$, that is, on the molecular model used for the gas-surface interaction. Satisfactory surface models have been developed recently for the calculation of velocity distributions for molecular beams scattered from crystalline surfaces.[5,6] Although it is not possible in general to express the collisional trajectory equations for these models in closed mathematical form, analytical collision equations can be obtained if the vibratory motion of the surface atoms is neglected. Experience with these models has shown that, in contrast with energy and normal momentum transfer, tangential momentum transfer rates at the surface are very insensitive to the extent of surface atom motion, and that, in fact, the atoms can be assumed to be completely immobile (i.e, either infinitely massive or rigidly held in the lattice) with very little error.§

As with most of the scattering models that have been studied, the surface model used here is two-dimensional in the sense that the gas-surface intermolecular potential depends only on two spatial coordinates—the coordinate normal to the surface and one surface coordinate. Specifically, the potential is assumed to depend on x, the coordinate parallel to the direction of the bulk flow, and y, the normal coordinate. Furthermore, the dependence on x is assumed to be periodic corresponding to the periodic arrangement of the surface atoms in the lattice. Since the periodicity of the potential will induce a periodicity in the distribution function, the quantities determined from Eq. (1) must be averaged over the period L in order to correspond to macroscopic quantities.

Table 1 Calculated values of surface quantities

l	V	n_s/n_0	C_s	C_D	f_1	F_1	σ_s	σ_d
0.25	0.001	1.00	0.991	1.23	0.990	0.997	0.018	0.00218
	0.01	1.00	0.991	0.123	0.990	0.997	0.018	0.00218
	0.10	1.00	0.991	0.0126	0.990	0.997	0.018	0.00223
	0.50	0.999	0.991	...	0.988	0.996	0.018	...
	1.00	0.993	0.991	...	0.983	0.992	0.018	...
0.50	0.001	1.00	0.962	35.16	0.979	0.989	0.076	0.0623
	0.01	1.00	0.962	3.516	0.979	0.989	0.076	0.0623
	0.10	1.00	0.962	0.352	0.979	0.988	0.076	0.0624
	0.50	0.993	0.963	0.0739	0.976	0.985	0.074	0.0655
	1.00	0.971	0.966	...	0.968	0.974	0.068	...
0.75	0.001	1.00	0.916	103.6	0.967	0.974	0.168	0.184
	0.01	1.00	0.916	10.36	0.967	0.974	0.168	0.184
	0.10	1.00	0.916	1.036	0.967	0.974	0.168	0.184
	0.50	0.986	0.920	0.208	0.965	0.968	0.160	0.184
	1.00	0.947	0.930	0.105	0.959	0.956	0.140	0.186
1.00	0.001	1.00	0.856	207.4	0.953	0.952	0.288	0.368
	0.01	1.00	0.856	20.74	0.953	0.952	0.288	0.368
	0.10	1.00	0.856	2.072	0.953	0.952	0.288	0.367
	0.50	0.979	0.865	0.406	0.952	0.948	0.270	0.360
	1.00	0.923	0.887	0.191	0.951	0.941	0.226	0.339
1.20	0.001	1.00	0.800	315.80	0.937	0.927	0.400	0.560
	0.01	1.00	0.800	31.58	0.937	0.927	0.400	0.560
	0.10	1.00	0.801	3.154	0.937	0.927	0.398	0.559
	0.50	0.975	0.814	0.611	0.940	0.928	0.372	0.541
	1.00	0.913	0.847	0.280	0.948	0.931	0.306	0.496
1.414	0.001	1.00	0.739	456.0	0.914	0.888	0.522	0.808
	0.01	1.00	0.739	45.60	0.914	0.888	0.522	0.808
	0.10	1.00	0.740	4.554	0.914	0.889	0.520	0.807
	0.50	0.972	0.757	0.881	0.922	0.897	0.486	0.781
	1.00	0.905	0.802	0.400	0.941	0.917	0.396	0.709

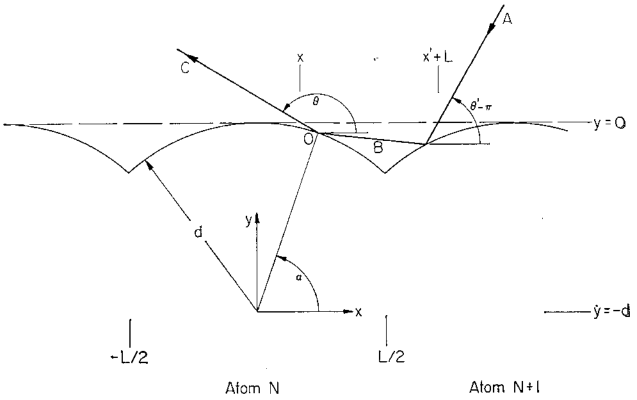

Fig. 1 Surface model.

With these assumptions about the model we obtain from Eqs. (1) and (2),

$$\mathbf{u}_s = \hat{\mathbf{x}}\left(\frac{n_0}{n_s}\right)\left(\frac{kT_0}{2m}\right)^{1/2}\left\{V + (\pi L)^{-1}e^{-V^2}\int_{-L/2}^{L/2}\int_0^{\pi}\cos\theta \times \left[B + \pi^{1/2}\left(\frac{1}{2} + B^2\right)e^{B^2}(1 + \operatorname{erf}B)\right]d\theta dx\right\} \tag{3a}$$

and

$$\tau_s \equiv (p_{yx})_s = n_0kT_0\left\{-\pi^{1/2}V + (\pi L)^{-1}e^{-V^2} \times \int_{-L/2}^{L/2}\int_0^{\pi}\sin\theta\cos\theta\left[B^2 + \frac{1}{2} + \pi^{1/2}B\left(\frac{3}{2} + B^2\right) \times e^{B^2}(1 + \operatorname{erf}B)\right]d\theta dx\right\} \tag{3b}$$

where $V = (n/2kT_0)^{1/2}U$, $B = V\cos\theta'(x,\theta)$, and θ and θ' are the cylindrical polar angles of the velocity vectors $\mathbf{c}$ and $\mathbf{c}'$. We also find that

$$n_s = \frac{1}{2}n_0\left\{1 + (\pi L)^{-1}e^{-V^2}\int_{-L/2}^{L/2}\int_0^{\pi}[1 + \pi^{1/2}Be^{B^2} \times (1 + \operatorname{erf}B)]d\theta dx\right\} \leq n_0$$

and that the surface temperature and normal heat flux satisfy $T_s \geq T_0$ and $(q_y)_s = -\tau_s u_s$.

We have evaluated these integral expressions for the slip velocity and surface shear stress for impulsive molecule-surface interactions. For this case the gas molecules are rigid spheres (with diameter d_g) and the surface is a two-dimensional array of immobile atoms of diameter d_s separated by a distance L in the x direction. A cross section of the model is shown in Fig. 1. The region that cannot be penetrated by the centers of the gas molecules is bounded by a series of parallel cylindrical segments of diameter $d = \frac{1}{2}(d_g + d_s)$. The plane of the "macroscopic" surface $y = 0$ is that tangent to the cylindrical surfaces, so the centers of the surface atoms lie on the plane $y = -d$. Because the interactions are impulsive and the surface atoms fixed, the collision trajectories are straight line segments terminating at impact points for which the angles of incidence and reflection measured from the local surface normal are equal (an example is the trajectory ABC shown in Fig. 1). Thus the collision function

---

§ For example, Goodman[5] found that the tangential momentum flux changes only a few percent as the mass of the surface atoms varies from 0 to ∞, while the energy and normal momentum fluxes change by more than a factor of ten. McClure[6] found for a typical gas-surface system with a gas temperature of 300°K that changing the surface atom mass by a factor of 3 produces a 0.1% change in the tangential momentum transfer rate, while the normal momentum and energy transfer rates change by 6.9% and 12.6%, respectively.

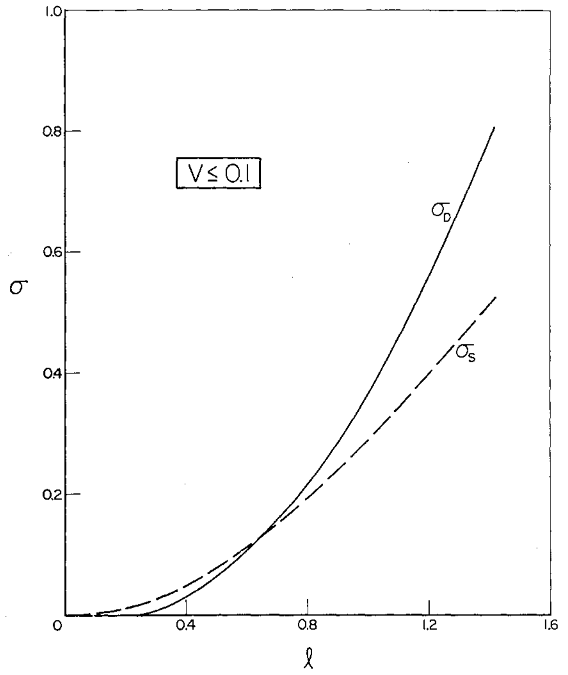

Fig. 2 Dependence of tangential momentum accommodation coefficient on l for $V \leq 0.1$.

$\theta'(x,\theta)$ can be determined analytically from the trajectory geometry for given values of the size ratio $l \equiv L/d$.

The evaluation of the slip velocity and surface shear stress was done numerically using an IBM 7040/7094 computer, and the results are given in Table 1 and Fig. 2 for thirty $l - V$ pairs. The slip velocity and shear stress are presented in terms of the slip coefficient $C_s = u_s/U$ and the drag coefficient $C_D = \tau_s/\frac{1}{2}\rho_0 U^2$. The tangential momentum accommodation coefficient σ is related to C_s and C_D by[7] $\sigma \equiv \sigma_s = 2(1 - C_s)$ and $\sigma \equiv \sigma_D = \pi^{1/2} V C_D$. Finally, as shown in Fig. 1, a given collisional trajectory can involve either one or two impact points, depending on the values of x, θ and l [for $l \leq (2)^{1/2}$]. The integrals in Eqs. (3) were separated into contributions from the two cases, and f_1, the fraction of *molecules* which undergo only one impact, and F_1, the fraction of the *collision rate* involving one impact, were computed.

The significance of the trends in the results is generally evident from Table 1, but a few observations seem worthy of emphasis. For low plate velocities, i.e., for V between 0.001 and 0.1, the slip coefficient and number density are independent of V for a given l, while the drag coefficient is inversely proportional to V. We note that this range of V corresponds to plate velocities up to about 100 fps for gases at room temperature. It must be remembered, however, that the results are applicable to free molecule flow only, so that the effect of plate speed could be quite different at higher densities. Figure 2 shows the variation of the tangential momentum accommodation coefficient with l for $V \leq 0.1$. One interesting result is the disparity between the values of σ_s and σ_D. It is a common practice to determine the value of σ from a measured value of either C_s or C_D and then to use this value to calculate the other coefficient. The results of our analysis indicate that this procedure, which amounts to assuming that $\sigma_s = \sigma_D$, can lead to appreciable errors, especially if $L > d$.

Finally, the comparison between the calculated and experimental values of σ with the parameter l is physically reasonable for all cases investigated, although the calculated magnitudes of σ are generally lower than those found experimentally. Experimental values in excess of 0.9 have been found for a variety of surfaces,[1,2] although values as low as 0.6 have been reported.[3] By contrast, the calculated values of σ_D vary from 0.22 to 0.56 in the range $l = 0.8 - 1.2$ typical of most surfaces. This disparity is probably the result of several factors. First, there is the experimental difficulty, typified by the wide range of reported values for σ, of accurate control and determination of the state of the surface. Second, there is the theoretical idealization made here concerning the two-dimensional, single-crystal nature of the surface. The consequences of this assumption, unlike those of the immobility of the surface atoms, have not been discussed in detail in the literature. That the calculated accommodation coefficients are low relative to typical measured values suggests that the polycrystalline nature of real surfaces has an appreciable effect on tangential momentum transfer at gas-solid interfaces.

References

[1] Schaaf, S. A. and Chambré, P. L., "Flow of Rarefied Gases," *High Speed Aerodynamics and Jet Propulsion*, Vol. III, Princeton University Press, Princeton N. J., 1958, pp. 687–739.

[2] Hurlbut, F. C., "On the Molecular Interactions between Gases and Solids," HE-150-208, Nov. 1962, Institute for Engineering Research, Univ. of California, Berkeley, Calif.

[3] Hurlbut, F. C., "Influence of Pressure History on Momentum Transfer in Rarefied Gas Flows," *The Physics of Fluids*, Vol. 3, No. 4, July-Aug. 1960, pp. 541–544.

[4] Hurlbut, F. C., "Current Developments in the Study of Gas-Surface Interactions," *Rarefied Gas Dynamics, Fifth International Symposium*, Vol. 1, Academic Press, New York, 1967, pp. 1–34.

[5] Goodman, F. O., "Preliminary Results of a Three-Dimensional Hard-Spheres Theory of Scattering of Gas Atoms from a Solid Surface," *Rarefied Gas Dynamics, Fifth International Symposium*, Vol. 1, Academic Press, New York, 1967, pp. 35–48.

[6] McClure, J. D., "Atomic and Molecular Scattering from Solids: II. Comparison of Classical Scattering Models in Relation to Experiment," *Journal of Chemical Physics*, Vol. 51, No. 5, Sept. 1969, pp. 1687–1700.

[7] Yang, H. T. and Lees, L., "Rayleigh's Problem at Low Reynolds Number According to the Kinetic Theory of Gases," *Rarefied Gas Dynamics, First International Symposium*, Pergamon Press, London, 1958, pp. 201–238.

Relationship between Temperature and Velocity Profiles in a Turbulent Boundary Layer along a Supersonic Nozzle with Heat Transfer

L. H. BACK* AND R. F. CUFFEL†
Jet Propulsion Laboratory, Pasadena, Calif.

Nomenclature

c_p = specific heat
c_f = friction coefficient, $(c_f)/(2) = (\tau)/(\rho_e u_e^2)$
H = enthalpy
M = Mach number
p = pressure
q = heat flux to wall
r = nozzle radius
r_c = throat radius of curvature
r_{th} = throat radius
St = Stanton number, $q/(H_{aw} - H_w)\rho_e u_e$
T = temperature
u = velocity parallel to wall

Received June 8, 1970. This work presents the results of one phase of research carried out in the Propulsion Research and Advanced Concepts Section of the Jet Propulsion Laboratory, California Institute of Technology, under Contract NAS7-100, sponsored by NASA.

* Member Technical Staff. Associate Fellow AIAA.

† Senior Engineer. Member AIAA.